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# The Discrete Gaussian Chain with $1/r^n$ Interactions: Exact Results

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We show that the discrete Gaussian chain with interaction $V(r) = 1/(r^2 - 1/4)$ is self-dual. At the dual temperature $k_B T = 1$ we calculate the height–height correlation function and find that the system is rough. A duality relation is established for the temperature-dependent correlation function exponent $\eta$. We also consider interactions $V(r) \simeq 1/r^n$ and show that absence of a phase transition for $2 < n < 3$ implies absence of a phase transition for $1 < n < 2$. All these results have their counterparts in a linear system of charges interacting through a potential which is asymptotically logarithmic (for $n = 2$) or power-law-like (for $n \neq 2$).

**KEY WORDS:** Discrete Gaussian model; long-range interactions; self-duality.

## 1. INTRODUCTION

One-dimensional systems having interactions that decay as $1/r^n$ with distance are of particular theoretical interest. Among them, the Ising model is rigorously known[1,2] to exhibit long-range order at sufficiently low temperature for $n < 2$, and not to have a phase transition for $n > 2$. The $1/r^2$ Ising model is a borderline case and has received special attention.[3–7] Its behavior can be analyzed in terms of topological defects which interact logarithmically. In this respect the model is similar to the two-dimensional Coulomb gas, which in turn is connected to the two-dimensional $XY$ model[8] and the two-dimensional discrete Gaussian model[9] (both with nearest-neighbor interactions).

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Recently Cardy[10] has given a renormalization group analysis of the $1/r^2$ interaction in one-dimensional systems whose site variables have access to a finite number of states with arbitrary symmetry. The renormalization equations for these models indicate a Kosterlitz–Thouless type of transition[8] with a correlation length that diverges as the exponential of a power when $T \downarrow T_c$. The examples presented in Ref. 10 are the $q$-state Potts model and the Ashkin–Teller model.

In this paper we present a number of exact analytic results on a closely related system, namely, the *discrete Gaussian* (DG) chain with interactions $V(r) \simeq 1/r^n$. We are able to derive our results by combining, in this new context, two transformations that occur in the literature. Firstly we map the DG chain onto a neutral gas of charges via a transformation taken from Chui and Weeks.[9] Secondly, we reconvert the gas of charges into a new discrete Gaussian chain with potential $V'(r) \simeq 1/r^{4-n}$ by Cardy's transformation.[10] Since the first transformation interchanges high and low temperatures, but the second one does not, the net result is an inversion of temperature. It follows that absence of a phase transition for $2 < n < 3$ implies absence of a phase transition for $1 < n < 2$. In the first case the DG chain, viewed as a model for an interface, is rough at all temperatures, and in the second case it is smooth at all temperatures.

For $n = 2$ one sees that $V(r)$ and $V'(r)$ have the same large-$r$ behavior, and the possibility of self-dual potentials arises. Indeed we find that the special potential $V^*(r) = 1/(r^2 - \frac{1}{4})$ is self-dual. The dual temperature is $k_B T = 1$. (The heuristic Kosterlitz–Thouless argument[8] predicts just this value as the critical temperature of the system!) For the potential $V^*(r)$ we present the following results:

With the aid of the unusually strong duality properties we calculate, at $k_B T = 1$, the height–height correlation function. It diverges logarithmically with distance, i.e., the system is rough at this temperature. A duality relation is derived for the temperature-dependent correlation function exponent $\eta$. If it is assumed that the dual temperature $k_B T = 1$ marks the transition between a smooth and a rough phase, then it follows that $\eta = 1$ for $k_B T < 1$ and that $\eta$ takes the same values as in the continuum Gaussian model for $k_B T > 1$. At the dual point itself we have—whether there is a transition or not—that $\eta = 2$, in contrast to the continuum Gaussian value $\eta = 3$.

## 2. THE DISCRETE GAUSSIAN MODEL

We consider a one-dimensional system of $N$ sites labeled $i = 1, 2, \ldots, N$. At each site there is a *height* variable $h_i$ taking the values

$0, \pm 1, \pm 2, \ldots$ . We take the periodic boundary condition $h_{i+N} = h_i$. The discrete Gaussian (DG) Hamiltonian $\mathcal{H}_N^{\mathrm{DG}}$ is defined by

$$\mathcal{H}_N^{\mathrm{DG}} = (1/2) \sum_{i \neq j} V_N(i-j)(h_i - h_j)^2 \tag{2.1}$$

The interaction $V_N(r)$ is chosen to satisfy

$$V_N(r) = V_N(-r), \qquad V_N(r) = V_N(r+N) \tag{2.2}$$

and we denote

$$V(r) = \lim_{N \to \infty} V_N(r) \tag{2.3}$$

The value of $V_N(0)$ is irrelevant and we arbitrarily set $V_N(0) = 0$. Specific choices for $V_N(r)$ will be made later.

We shall consider the partition function $Z_N^{\mathrm{DG}}$ defined by

$$Z_N^{\mathrm{DG}}[\beta V_N] = \sum_{\{h_i\}}{}' e^{-\beta \mathcal{H}_N^{\mathrm{DG}}} \tag{2.4}$$

where $\beta \equiv (k_B T^{\mathrm{DG}})^{-1}$ is the inverse temperature. The prime on the summation sign indicates that the height variable $h_N$ is kept at the fixed value $h_N = 0$: in this way we avoid that $Z_N^{\mathrm{DG}}$ becomes infinite due to the invariance of $\mathcal{H}_N^{\mathrm{DG}}$ under $\{h_i\} \to \{h_i + m\}$ (where $m$ is any integer). This condition on the summation in (2.4) will be seen to play an important role.

In our later discussion we shall need the Fourier transforms $\hat{V}_N(k)$ and $\hat{h}_k$ defined by

$$h_j = N^{-1/2} \sum_k e^{ikj} \hat{h}_k \tag{2.5}$$

$$V_N(r) = N^{-1} \sum_k e^{ikr} \hat{V}_N(k) \tag{2.6}$$

where the sum is on the wave numbers $k = 0, \pm 2\pi/N, \pm 4\pi/N, \ldots, \pm(N-1)\pi/N$ (for $N$ odd) or $k = 0, \pm 2\pi/N, \pm 4\pi/N, \ldots, \pm(N-2)\pi/N, \pi$ (for $N$ even). The Hamiltonian takes the form

$$\mathcal{H}_N^{\mathrm{DG}} = \sum_k W_N(k) \hat{h}_k \hat{h}_{-k} \tag{2.7}$$

where

$$W_N(k) = \hat{V}_N(0) - \hat{V}_N(k) \tag{2.8}$$

Clearly we have to impose the condition

$$W_N(k) > 0 \qquad \text{for} \quad k \neq 0 \tag{2.9}$$

to keep $Z_N^{\mathrm{DG}}$ from blowing up.

## 3. EQUIVALENCE TO A SYSTEM OF INTERACTING CHARGES AND SELF-DUALITY

### 3.1. Chui and Weeks' Transformation

In this section we apply to $Z_N^{\rm DG}$ a transformation used by Chui and Weeks[9] for the nearest-neighbor discrete Gaussian model. The transformation can in fact be used for interactions of arbitrary range and was employed by other authors in various different contexts.[11,12] For the present application it is essential that we take account of the subtle effects[13,14] due to the finiteness of the system and the condition $h_N = 0$ on the summation in (2.4). We recall here only the main steps and refer to Appendix A for details.

The transformation consists in using Poisson's summation formula in the expression (2.4). This replaces the summation on the integer-valued heights $h_i$ by integrations on continuous variables $\nu_i$ and summations on new integer-valued variables $q_i$ (called *charges*; $q_i = 0, \pm 1, \pm 2, \ldots$). Fourier transforms $\hat{\nu}_k$ and $\hat{q}_k$ are defined as in (2.5). The partition function then takes the form

$$Z_N^{\rm DG}[\beta V_N] = \sum_{\{q_i\}} \int_{-1/2}^{1/2} d\lambda \int_{-\infty}^{\infty} \prod_k d\hat{\nu}_k \times \exp\left[2\pi i \sum_k (N^{-1/2}\lambda + \hat{q}_{-k})\hat{\nu}_k - \beta \sum_k W_N(k)\hat{\nu}_k \hat{\nu}_{-k}\right] \tag{3.1}$$

Here the occurrence of the $\lambda$ integration is a consequence of the fact that the sum in (2.4) is restricted to $h_N = 0$. When carrying out, in (3.1), the integrations on $\hat{\nu}_0$ and $\lambda$ we find the "charge neutrality condition" $\sum_{i=1}^N q_i = 0$. Doing the remaining $\hat{\nu}_k$ integrations we find

$$Z_N^{\rm DG}[\beta V_N] = N^{1/2} C_N^{1/2} \beta^{-(N-1)/2} \sum_{\{q_i\}}{}'' \exp\left[-\frac{\pi^2}{\beta} \sum_{k \neq 0} \frac{\hat{q}_k \hat{q}_{-k}}{W_N(k)}\right] \tag{3.2}$$

where the double prime denotes the charge neutrality condition and where

$$C_N = \prod_{k \neq 0} \frac{\pi}{W_N(k)} \tag{3.3}$$

Reconverting the $\hat{q}_k$ in (3.2) to $q_i$ we obtain

$$Z_N^{\rm DG}[\beta V_N] = N^{1/2} C_N^{1/2} \beta^{-(N-1)/2} Z_N^{\rm CG}\left[\frac{1}{\beta} U_N\right] \tag{3.4}$$

in which $Z_N^{\rm CG}$ is the partition function for a gas of charges (CG) described by a Hamiltonian $\mathcal{H}_N^{CG}$:

$$Z_N^{CG}\left[\frac{1}{\beta}U_N\right] = \sum_{\{q_i\}}'' \exp\left(-\frac{1}{\beta}\mathcal{H}_N^{CG}\right) \tag{3.5}$$

$$\mathcal{H}_N^{CG} = -(1/2)\sum_{i\neq j} U_N(i-j)q_iq_j \tag{3.6}$$

in which the charge potential is given by

$$U_N(r) = \frac{2\pi^2}{N}\sum_{k\neq 0}\frac{1-\cos kr}{W_N(k)}, \qquad r = 1,2,\ldots,N-1 \tag{3.7}$$

The interaction $U_N(r)$, like $V_N(r)$, is symmetric and $N$ periodic in $r$. We shall set $U_N(0) = 0$. Furthermore, we denote

$$U(r) = \lim_{N\to\infty} U_N(r) \tag{3.8}$$

Equation (3.4) relates the partition function of the original DG model with interaction $V_N(r)$ to the partition function of a gas of charges with potential $U_N(r)$. The latter is at inverse temperature $(k_B T^{CG})^{-1} = 1/\beta$, and hence (3.4) connects high $T^{DG}$ to low $T^{CG}$ and vice versa. The equations (2.6), (2.8), and (3.7) give the connection between $U_N(r)$ and $V_N(r)$.

In Appendix B we show that in the limit $N\to\infty$ the long-distance behaviors of $U(r)$ and $V(r)$ are related as follows. If

$$V(r)\sim 1/r^n, \qquad r\to\infty, \qquad 1<n<3 \tag{3.9}$$

then

$$W(k)\sim k^{n-1}, \qquad k\to 0 \tag{3.10}$$

and

$$U(r)\sim\begin{cases} r^{n-2}, & r\to\infty, \quad 2<n<3 & (3.11a)\\ \log r, & r\to\infty, \quad n=2 & (3.11b)\\ u_0 - u_1/r^{2-n}, & r\to\infty, \quad 1<n<2 & (3.11c)\end{cases}$$

where $u_0$ and $u_1$ are positive constants.

### 3.2. Cardy's Transformation

In this section we transform the partition function $Z_N^{CG}[(1/\beta)U_N]$ of equation (3.5) with the aid of a transformation introduced by Cardy.[10] Cardy's purpose was to express the Hamiltonian of a one-dimensional system with site variables taking a finite number of values, and having $1/r^2$ interaction between themselves, in terms of kink variables. Application of the transformation here, in the reversed sense, to the charges $q_i$ simply amounts to the introduction of new variables $l_1, l_2, \ldots, l_N$ defined by

$$l_i = -\sum_{j=i}^{N-1} q_j, \qquad i = 1, 2, \ldots, N-1 \tag{3.12a}$$

$$l_N = 0 \tag{3.12b}$$

The neutrality condition on the charges $q_i$ is automatically satisfied and the $l_i$ (for $i \neq N$) take independently all integer values. This means that in (3.5) we may replace the summation $\sum''$ on the $q_i$ by a summation $\sum'$ on the $l_i$, where the prime has the same meaning as before. In this way (3.5), (3.6), and (3.12) lead to an expression for $Z_N^{\mathrm{CG}}$ as a new discrete Gaussian partition function at the same temperature:

$$Z_N^{\mathrm{CG}}\left[\frac{1}{\beta} U_N\right] = \sum_{\{l_i\}}{}' \exp\left(-\frac{1}{\beta}\mathcal{H}_N'^{\mathrm{DG}}\right) = Z_N^{\mathrm{DG}}\left[\frac{1}{\beta} V_N'\right] \tag{3.13}$$

with

$$\mathcal{H}_N'^{\mathrm{DG}} = (1/2)\sum_{i \neq j} V_N'(i-j)(l_i - l_j)^2 \tag{3.14}$$

in which the new interaction $V_N'(r)$ is given by

$$V_N'(r) = -\tfrac{1}{2}\left[U_N(r-1) - 2U_N(r) + U_N(r+1)\right], \qquad r = 1, 2, \ldots, N-1 \tag{3.15}$$

$V_N'(r)$ is symmetric and $N$ periodic. We set $V_N'(0) = 0$, and define $V'(r) = \lim_{N\to\infty} V_N'(r)$. Equation (3.15) shows that if $U(r)$ behaves as in (3.11) for large $r$, then

$$V'(r) \sim 1/r^{4-n} \qquad r \to \infty \tag{3.16}$$

### 3.3. Duality and Self-Duality

From Eqs. (3.4) and (3.13) we see that the partition functions of the two DG models with potentials $V_N$ and $V_N'$ are related by

$$Z_N^{\mathrm{DG}}[\beta V_N] = N^{1/2} C_N^{1/2} \beta^{-(N-1)/2} Z_N^{\mathrm{DG}}\left[\frac{1}{\beta} V_N'\right] \tag{3.17}$$

The connection between $V_N(r)$ and $V_N'(r)$ is most conveniently expressed in terms of their Fourier transforms $W_N(k)$ and $W_N'(k)$. From (2.8), (3.7), and (3.15) we find the *duality relation*

$$W_N'(k) = \frac{4\pi^2 \sin^2 \frac{1}{2}k}{W_N(k)} \tag{3.18}$$

It follows that there is a *self-dual* potential given by

$$W_N^*(k) = 2\pi|\sin \tfrac{1}{2}k| \tag{3.19}$$

We shall indicate all quantities referring to this potential by an asterisk. By setting $\beta = 1$ in (3.17) we find that we must have

$$NC_N^* = 1 \tag{3.20}$$

as may also be verified explicitly from (3.3) and (3.19). Letting $\beta$ again be general in (3.17) we obtain

$$Z_N^{\mathrm{DG}}[\beta V_N^*] = \beta^{-(N-1)/2} Z_N^{\mathrm{DG}}\left[\frac{1}{\beta} V_N^*\right] \tag{3.21}$$

which shows that the DG chain with interaction (3.19) has the *dual temperature*

$$\beta = 1 \tag{3.22}$$

The spatial representation of the self-dual interaction $W_N^*(k)$ is readily obtained as

$$V_N^*(r) = \frac{(\pi/N)\sin(\pi/N)}{\sin[(\pi/N)(r+\frac{1}{2})]\sin[(\pi/N)(r-\frac{1}{2})]}, \qquad r = 1, \ldots, N-1 \tag{3.23}$$

whence

$$V^*(r) = \frac{1}{r^2 - \frac{1}{4}}, \qquad r = \pm 1, \pm 2, \ldots \tag{3.24}$$

The charge potential in the equivalent CG model is

$$U_N^*(r) = \frac{2\pi}{N} \sum_{s=1}^{r} \cot \frac{\pi}{N}(s - \tfrac{1}{2}), \qquad r = 1, \ldots, N-1 \tag{3.25}$$

whence

$$U^*(r) = 2 \sum_{s=1}^{|r|} \frac{1}{s - \frac{1}{2}}, \qquad r = \pm 1, \pm 2, \ldots \tag{3.26}$$

which for $r \to \infty$ behaves as $U^*(r) \simeq 2 \log r + O(1)$. Hence we have found a DG chain with interaction $\simeq 1/r^2$, and a gas of charges with interaction $\simeq 2\log r$, that each satisfy a high-temperature–low-temperature duality and have $\beta = 1$ as their dual point.

### 3.4. Absence of Phase Transitions for $n \neq 2$

One-dimensional models with interactions decaying as $1/r^n$, where $n > 2$, are generally expected not to exhibit a phase transition.[1,5] There seems to be no reason why this would not also hold for the DG chain with $V(r) \simeq 1/r^n$ and $n > 2$. We saw in the previous section [Eqs. (3.9) and (3.16)], that the potential $V(r) \simeq 1/r^n$ is dual to the case $V(r) \simeq 1/r^{4-n}$. Hence, by (3.17), absence of a phase transition for $2 < n < 3$ in the DG chain implies that there is no phase transition either in this chain for $1 < n < 2$. We conclude that the DG chain with interaction $1/r^n$, while rough at all temperatures when $n > 2$, is smooth at all temperatures for $n < 2$. The latter result is surprising in that it contrasts with the Ising chain, which does have a phase transition for $n < 2$. For the equivalent system of charges we obtain the corresponding statements by introducing a potential $U'(r)$ related to $V'(r)$ in the same way as $U(r)$ is related to $V(r)$. Comparison of $U'(r)$ and $U(r)$ shows that potentials $U(r) \simeq u_0 - u_1 r^m$ have no phase transition when $-1 < m < 0$ (plasma phase at all temperatures) or when $0 < m < 1$ (dielectric or dipole phase at all temperatures).

These considerations bring out that the potentials $V(r) \simeq 1/r^2$ [or $U(r) \simeq 2\log r$] are a borderline case. It is the only case where a phase transition may occur. We shall discuss the special self-dual potential $V^*(r) = 1/(r^2 - \frac{1}{4})$ in greater detail in the next section.

## 4. HEIGHT–HEIGHT CORRELATIONS FOR $V(r) = 1/(r^2 - 1/4)$

### 4.1. The Correlation Function at the Dual Point $\beta = 1$

The duality expressed by Eqs. (3.17) and (3.18) is unusually strong. For $V_N(r) = V_N^*(r)$ it enables us to find interesting relations between the high-temperature and the low-temperature behavior of the height–height correlation function, and it allows us in particular to fully calculate this quantity at $\beta = 1$. To see this we differentiate (3.17) with respect to $W_N(k)$ and put $W_N = W_N^*$. This gives

$$\beta \langle \hat{h}_k \hat{h}_{-k} \rangle_\beta^* + \frac{1}{\beta} \langle \hat{h}_k \hat{h}_{-k} \rangle_{1/\beta}^* = \frac{1}{2W_N^*(k)} \tag{4.1}$$

Setting $\beta = 1$ we obtain the correlation function at the dual point,

$$\langle \hat{h}_k \hat{h}_{-k} \rangle_1^* = \frac{1}{4W_N^*(k)} \tag{4.2}$$

From (2.7) and (4.2) we find for the internal energy of the system

$$\langle \mathcal{H}_N^{\mathrm{DG}} \rangle_1^* = \tfrac{1}{4}(N-1) \tag{4.3}$$

while Fourier transformation of (4.2) gives the height–height correlation as

$$\left\langle (h_i - h_{i+r})^2 \right\rangle_1^* = \frac{2}{N} \sum_{k \neq 0} (1 - \cos kr) \langle \hat{h}_k \hat{h}_{-k} \rangle_1^*$$

$$= \frac{1}{4\pi^2} U_N^*(r) \tag{4.4}$$

By (3.26) we have that in the thermodynamic limit $N \to \infty$

$$\left\langle (h_i - h_{i+r})^2 \right\rangle_1^* \simeq \frac{1}{2\pi^2} \log r, \qquad r \to \infty \tag{4.5}$$

which means that at the dual point the DG system is rough.

The exponent $\eta(\beta)$ which is usually associated with a DG model is defined by

$$g(r;\beta) \equiv \exp\left[ -2\pi^2 \left\langle (h_i - h_{i+r})^2 \right\rangle_\beta \right] \sim 1/r^{\eta(\beta)-1} \tag{4.6}$$

(where we have taken into account that our system has dimension 1). For $\eta = 1$ the height fluctuations $\langle (h_i - h_{i+r})^2 \rangle_\beta$ tend to a finite limit as $r \to \infty$ and the interface is smooth; for any $\eta > 1$ it is rough. From (4.5) and (4.6) we see that for $V = V^*$ we have

$$g^*(r;1) \sim 1/r \tag{4.7a}$$

$$\eta(1) = 2 \tag{4.7b}$$

It is interesting to remark that the analog of (4.7b) in the continuum Gaussian model[3] is that $\eta(1) = 3$, implying a rougher interface than in the DG model.

## 4.2. The Correlation Function for General $\beta$

Duality arguments do not permit to extend the exact result (4.4) for the height–height correlation to general $\beta$. However, we can derive interesting relations from the duality property (4.1). Since the large-$r$ behavior of $g(r;\beta)$ is determined by the small $k$ behavior of $\langle \hat{h}_k \hat{h}_{-k} \rangle_\beta$ we put

$$\langle \hat{h}_k \hat{h}_{-k} \rangle_\beta^* \simeq c(\beta) k^{\sigma(\beta)}, \qquad \text{as} \quad k \to 0 \tag{4.8}$$

where $c(\beta)$ and $\sigma(\beta)$ are unknown. A calculation analogous to the one of the preceding subsection shows that $\eta(\beta) = 1 + 4\pi c(\beta)$ if $\sigma(\beta) = -1$ and

[3] For the continuum Gaussian model with the same interaction $V_N^*(r)$ one shows in a straightforward manner that $\langle \hat{h}_k \hat{h}_{-k} \rangle_\beta^* = 1/2\beta W_N^*(k)$ at all $\beta$.

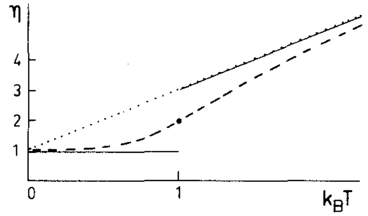


Fig. 1. The exponent $\eta$ as a function of the temperature $T \equiv T^{\mathrm{DG}}$ of the discrete Gaussian model. Dashed line: a possible solution of (4.9) if there is no phase transition. Solid line with isolated point at $k_B T = 1$: the solution (4.10) of (4.9) if the DG model has a smooth low-temperature phase for $k_B T < 1$. For comparison the dotted line gives the exponent $\eta$ of the continuum Gaussian model.

$\eta(\beta) = 1$ if $\sigma(\beta) > -1$. Upon using (3.19) and (4.8) in (4.1) and considering the leading terms for $k \to 0$ we find for the exponent $\eta(\beta)$ the duality relation

$$\beta\eta(\beta) + \beta^{-1}\eta(\beta^{-1}) = 2 + \beta + \beta^{-1} \tag{4.9}$$

of which (4.7b) is a special case.

We cannot rule out the possibility that the DG chain with potential $V^*(r) = 1/(r^2 - \frac{1}{4})$ has no phase transition, in which case (4.9) would be satisfied by some analytic $\eta(\beta)$, as in Fig. 1. If one assumes, however, that the system is in a smooth phase for $k_B T^{\mathrm{DG}} < 1$ $(\beta > 1)$, then we have the interesting behavior

$$\eta(\beta) = \begin{cases} 1, & \beta > 1 \\ 2, & \beta = 1 \\ 1 + 2/\beta, & \beta < 1 \end{cases} \tag{4.10}$$

which is also shown in Fig. 1. It has the extraordinary feature that above criticality ($\beta < 1$) the exponent $\eta(\beta)$ has precisely the same value as in the continuum Gaussian model (see footnote 3). At criticality $\eta$ gets renormalized to a value below the continuum Gaussian value, which yields an isolated point on the $\eta(\beta)$ curve.

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## APPENDIX A

We give here some details on the transformation leading from (2.4) to (3.1)–(3.3). Poisson's summation formula reads[15]

$$\sum_{h_j=-\infty}^{\infty} f(h_j) = \int_{-\infty}^{\infty} d\nu_j \sum_{q_j=-\infty}^{\infty} \exp(2\pi i q_j \nu_j) f(\nu_j) \tag{A1}$$

Since the sum in (2.4) is subject to the condition $h_N = 0$ we also use

$$\begin{aligned} f(h_N) \equiv f(0) &= \int_{-\infty}^{\infty} d\nu_N\, \delta(\nu_N) f(\nu_N) \\ &= \int_{-\infty}^{\infty} d\nu_N \sum_{q_N=-\infty}^{\infty} \int_{-1/2}^{1/2} d\lambda \exp\left[2\pi i \nu_N (q_N + \lambda)\right] f(\nu_N) \end{aligned} \tag{A2}$$

in which the last equality follows from the integral representation of the $\delta$ function. We obtain Eq. (3.1) by inserting (A1) and (A2) in (2.4) and using (2.7). In (3.1) the integrations on $\hat{\nu}_0$ and $\lambda$ are special. Since $W_N(0) = 0$ they read

$$\int_{-1/2}^{1/2} d\lambda \int_{-\infty}^{\infty} d\hat{\nu}_0 \exp\left[2\pi i \left(N^{-1/2}\lambda + \hat{q}_0\right)\hat{\nu}_0\right]$$

$$= \int_{-1/2}^{1/2} d\lambda\, N^{1/2} \delta\left(\lambda + \sum_{j=1}^{N} q_j\right) \tag{A3}$$

$$= N^{1/2} \delta_{0, \sum_{j=1}^{N} q_j} \tag{A4}$$

The remaining $\hat{\nu}_k$ integrations in (3.1) are standard Gaussian integrals which one can perform (doing the real and the imaginary parts of the $\hat{\nu}_k$ separately) with the aid of the formula

$$\int_{-\infty}^{\infty} dx\, e^{-ax^2 + ibx} = \left(\frac{\pi}{a}\right)^{1/2} e^{-b^2/4a} \tag{A5}$$

As a result (3.1) is converted into (3.2).

## APPENDIX B

In order to obtain the large-$r$ behavior of $U(r)$ we combine Eqs. (3.7) and (3.8) and write

$$U(r) = 2\pi \int_0^{\pi} \frac{1 - \cos kr}{W(k)}\, dk \tag{B1}$$

If $V(r) \sim 1/r^n$ for $r \to \infty$, then $W(k) \sim w_0 k^{n-1} + o(k^{n-1})$ for $k \downarrow 0$, and the integral converges for all $n$ in the interval of interest, viz., $1 < n < 3$.

For $2 \leqslant n < 3$, the function $1/W(k)$ is itself nonintegrable at $k = 0$, and therefore $U(r)$ diverges as $r \to \infty$. To find the divergence we put $kr = x$ and obtain from (B1):

$$U(r) = 2\pi r^{n-2} \int_0^{\pi r} \frac{1 - \cos x}{w_0 x^{n-1} + \cdots} dx \tag{B2}$$

where the dots indicate terms that vanish as $r \to \infty$. Furthermore, in this limit the upper integration bound $\pi r$ may be replaced with $\infty$ and the integral becomes a constant. Hence (B2) implies the behavior (3.11a, b).

For $1 < n < 2$, the function $1/W(k)$ is itself integrable. In this case we split (B1) up into two terms. After applying a partial integration to the second one we find

$$U(r) = 2\pi \int_0^{\pi} \frac{dk}{W(k)} + \frac{2\pi}{r} \int_0^{\pi} \sin kr \frac{d}{dk} \frac{1}{W(k)} dk \tag{B3}$$

The first term equals a constant $u_0$. The large-$r$ behavior of the second term is obtained by putting again $kr = x$ and proceeding as before. This leads to the behavior (3.11c).

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